Some Properties of Linear Systems Defined Over a Commutative Banach Algebra*

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ABSTRACT

We consider linear systems defined over a commutative Hermitian Banach * algebra. In this case, the problem about spectrum displacement posed by Kaashoek, van der Mee, and Rodman can be solved; results related to stability, stabilizability, and detectability for such systems are obtained, which are a natural extension of the corresponding ones in the finite dimensional case.

I. INTRODUCTION AND PRELIMINARIES

In recent years, there have appeared many papers studying linear systems over a commutative Banach algebra. Such systems arise in the study of long strings of coupled systems, such as strings of vehicles. They also result from the discretization of partial differential equations. An example of such a discretization is the representation for a long seismic cable used in offshore oil exploration. Also, they result from the linearization of nonlinear systems with respect to nominal operating points specified in terms of a set of parameters. An example is the satellite problem, which is linearized with respect to a nominal radius and nominal angular velocity see [6–12]. In contrast to the work on general linear infinite dimensional systems, we shall

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be able to utilize Gelfand transform techniques in the study of system behavior and in the study of control for systems defined over a commutative Banach algebra. Furthermore, if a commutative Banach algebra is Hermitian * algebra, then matrices defined over this algebra have many similar properties to matrices defined over a complex field; for example, we can define Hermitian matrices and positive definite matrices, and the spectrum of every Hermitian matrix is real. In fact, in [6–8], the authors actually study linear systems over a commutative Hermitian Banach * algebra. A lot of nice results about stability, stabilizability, observability, and controllability for such systems have been obtained.

Let us consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad t > 0, \\ x(0) &= x_0 \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \qquad x(0) = x_0, \end{aligned} \tag{1.2}$$

where $x \in X$, $u \in U$, $y \in Y$. For the finite dimensional case (i.e., X, U, Y are finite dimensional linear spaces), it is well known that [2–4, 19]:

(1) The observed subsystem (C, A) of (1.2) is observable iff for every symmetric complex set Λ containing n elements, there exists an $K \in L(Y, X)$ such that $\sigma(A + KC) = \Lambda$ or

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

is left invertible for every $\lambda \in \mathbb{C}$, i.e., the Hautus condition is satisfied.

(2) The control subsystem (A, B) of (1.2) is controllable iff for every symmetric complex set Λ containing n elements, there exists an $F \in L(X, U)$ such that $\sigma(A + BF) = \Lambda$ or $[\lambda I - A \;\; B]$ is right invertible for every $\lambda \in \mathbb{C}$.

(3) The observed subsystem (C, A) of (1.2) is detectable iff $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ is left invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$.

(4) The control subsystem (A, B) of (1.2) is stabilizable iff $[\lambda I - A \;\; B]$ is right invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$.

(5) The system (1.1) is stable iff for every positive definite matrix Q, the Lyapunov equation $A^*P + PA = -Q$ has a positive definite solution.

For the infinite dimensional case, there are different concepts of controllability, observability, stability, stabilizability, and detectability; see [5, 7, 13–16]. When X, U, and Y are Hilbert spaces, Kaashoek et al. [1] and Takahashi [5] have respectively proved that:

1. (C, A) is exactly observable iff there exists an operator $K \in L(Y, X)$ such that $\sigma(A) \cap \sigma(A - KC) = \varnothing$; which is equivalent to either of the following equivalent statements:

   a. For some positive integer m, the operator

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \in L(X, Y^m)$$

   is left invertible, and K can be concretely constructed.

   b. For every $\lambda \in \mathbb{C}$, $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ is left invertible.

2. (A, B) is exactly controllable iff there exists an operator $F \in L(X, U)$ such that $\sigma(A) \cap \sigma(A + BF) = \varnothing$, which is equivalent to either of the following equivalent statements:

   c. For some positive integer m, the matrix $[B \; AB \; \cdots \; A^{m-1}B] \in L(U^m, X)$ is right invertible.

   d. For every $\lambda \in \mathbb{C}$, $[\lambda I - A \;\; B]$ is right invertible.

But when X, U, Y are Banach spaces, the authors of [1] indicate that whether the above-mentioned results are right or not is unknown.

It is obvious that both Hilbert spaces and commutative Hermitian Banach * algebras are Banach spaces, but they are not contained in each other. In this paper, it will be proved that the above-mentioned results are still right in commutative Hermitian Banach * algebras, so the problem posed by [1] is partially solved. As for stability, stabilizability, controllability, and observability, the authors of [6–8] have already given some characterizations. In this section we will give some new results on stability and stabilizability, and some on detectability. After Gelfand transformation, all results are reduced to the corresponding ones in finite dimensional systems.

Throughout this paper, let W denote a complex commutative Hermitian Banach * algebra with unit, that is, W as a commutative Banach algebra has a continuous involution $a \to a^*$, and the spectrum of every self-adjoint element is real [6–8, 18]. Given a positive integer n, we shall denote by W^n the complex Banach space of all n-element vectors with entries in W with norm

$$\|x\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{1/2},$$

where x_i is the ith component of the vector $x \in W^n$. Given positive integers (m, n), we define $M_{m\times n}(W)$ as the complex vector space consisting of all $m \times n$ matrices with entries in W, with natural addition and scalar multiplication, and with norm

$$\|A\|_2 = \sup\{\|Ax\|_2 : \|x\|_2 \leqslant 1,\ x \in W^n\},$$

where $A \in M_{m\times n}(W)$. In particular $M_n(W)$ is defined to be $M_{n\times n}(W)$; each element in $M_{m\times n}(W)$ can be thought of as a bounded linear operator from the Banach space W^m to W^n. Under this consideration, $M_n(W)$ is a unital Banach subalgebra of $L(W^n)$. Given $A \in M_n(W)$, $\sigma(A)$ is defined to be $\sigma_{M_n(W)}(A)$; $\sigma(A)$ represents the spectrum of A in $M_n(W)$. Given $F = (f_{ij}) \in M_{m\times n}(W)$, F^* is defined to be $F^* = (f_{ji}^*) \in M_{n\times m}(W)$; with the involution, $M_n(W)$ becomes an Hermitian Banach * algebra.

Given $A \in M_n(W)$, $B \in M_{n\times m}(W)$, $C \in M_{r\times n}(W)$, the matrix triple (A, B, C) determines a linear continuous time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \qquad (t > 0), \\ x(0) &= x_0 \in W^n, \end{aligned} \tag{1.3}$$

where $x(t)$, $u(t)$, $y(t)$ are locally Lebesgue-Bochner-integrable functions from $[0, \infty]$ to W^n, W^m, W^r respectively; we denote the control subsystem of the system (1.1) by (A, B) and the observed subsystem by (C, A).

DEFINITION 1.1. Let $A \in M_n(W)$, $B \in M_{n\times m}(W)$, $C \in M_{r\times n}(W)$. The control subsystem (A, B) is said to be exactly controllable if there exists a positive integer N such that $W^n = \sum_{i=1}^{N} A^i BW^m$. The control subsystem (A, B) is said to be approximately controllable if W^n is the linear closure of $\{A^k BW^m : k \geqslant 0\}$, in short, $W^n = \bigvee_{k=0}^{\infty} A^k BW^m$. The observed subsystem (C, A) is said to be exactly (approximately) observable if its dual control system (A^*, C^*) is exactly (approximately) controllable.

DEFINITION 1.2. The system $\dot{x}(t) = Ax(t)$, where $A \in M_n(W)$, is said to be stable if there exist a $\delta > 0$ and $M \geqslant 1$ such that

$$\|e^{At}\| \leqslant Me^{-\delta t}, \qquad \text{i.e.} \quad \sigma(A) \subset \mathbb{C}_-.$$

DEFINITION 1.3. The system (A, B) is said to be stabilizable if there exists an $F \in M_{m \times n}(W)$ such that the system

$$\dot{x}(t) = (A + BF)x(t)$$

is stable; i.e., there exist a $\delta > 0$ and $M \geqslant 1$ such that

$$\|e^{(A+BF)t}\| \leqslant Me^{-\delta t}.$$

DEFINITION 1.4. The system (C, A) is said to be detectable if there is a $G \in M_{n \times r}(W)$ such that

$$\dot{x}(t) = (A + GC)x(t)$$

is stable [5, 7, 14].

Let X denote the carrier (maximal ideal) space of W. Given $a \in W$, let $\hat{a}$ denote the Gelfand transform of a, which is a continuous function on the compact Hausdorff space X. Then for a matrix $F = (f_{ij}) \in M_{n \times n}(W)$, the Gelfand transform of F is defined to be $\hat{F} = (\hat{f}_{ij})$. It is easy to show that $\det F(\varphi) = \det(f_{ij}(\varphi))$ for any square matrix $F = (f_{ij})$ over W and for any $\varphi \in X$.

LEMMA 1.5. *Let $A \in M_n(W)$. Then*

$$\begin{aligned}\sigma(A) &= \bigcup \left\{\sigma\left(\hat{A}(\varphi)\right) : \varphi \in X\right\} \\ &= \bigcup_{\varphi \in X} \{\lambda_1(\varphi), \lambda_2(\varphi), \dots, \lambda_n(\varphi)\},\end{aligned}$$

where $\lambda_i(\varphi) \in \sigma(\hat{A}(\varphi)) = \{\lambda_1(\varphi), \lambda_2(\varphi), \dots, \lambda_n(\varphi)\}$ represents the set of eigenvalues of the $n \times n$ matrix $\hat{A}(\varphi)$.

By Lemma 1.5, the following facts are equivalent:

(1) $F \in M_n(W)$ is invertible.
(2) For each $\varphi \in X$, $\det(\hat{f}_{ij}(\varphi) \neq 0$.
(3) $\det F$ is invertible in W.

DEFINITION 1.6. A Hermitian element $V \in M_n(W)$ is called positive definite, and we write $V > 0$, if $\lambda > 0$ for every $\lambda \in \sigma(V)$.

LEMMA 1.7. *A Hermitian element* $V \in M_n(W)$ *is positive definite if and only if* $\hat{V}(\varphi)$ *is positive definite for all* $\varphi \in X$

II. SPECTRUM DISPLACEMENT WITH CONTROLLABILITY AND OBSERVABILITY

LEMMA 2.1 (Theorem 2.3 of [6]). *Given* $A \in M_n(W)$, $B \in M_{n \times m}(W)$, *the following conditions are equivalent*:

(1) *The control system* (A, B) *is exactly controllable.*

(2) *The control system* (A, B) *is approximately controllable.*

(3) *For each* $\varphi \in X$, *the finite dimensional linear system* $(\hat{A}(\varphi), \hat{B}(\varphi))$ *is controllable.*

(4) *For each complex number* λ, *there exists* $T_\lambda \in M_{(n+m)\times n}(W)$ *such that* $[\lambda I - A, B]T_\lambda = I_n$.

(5) *The operator* $[B \;\; AB \;\; \cdots \;\; A^{k-1}B]$ *is right invertible for some* k.

LEMMA 2.2 (Theorem 2.3 of [6]). *Given* $A \in M_n(W)$, $C \in M_{r \times n}(W)$, *the following conditions are equivalent*:

(1) *The observed system* (C, A) *is exactly observable.*

(2) *The observed system* (C, A) *is approximately observable.*

(3) *For each* $\varphi \in X$, *the finite dimensional linear system* $(\hat{C}(\varphi), \hat{A}(\varphi))$ *is observable.*

(4) *For each complex number* λ, *there exists* $H_\lambda \in M_{n \times (n+r)}(W)$ *such that*

$$H_\lambda \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = I_n.$$

(5) *The operator*

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix}$$

is left invertible for some m.

THEOREM 2.3 (Theorem 4.1 of [1]). *Assume G, H are Hilbert spaces; let $A \in L(G)$, $C \in L(G, H)$. Then there exists an operator $K \in L(H, G)$ such that $\sigma(A) \cap \sigma(A - KC) = \varnothing$ if and only if for some positive integer m the operator*

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \in L(G, H^m)$$

is left invertible. Moreover, in that case, given an open disc $\Delta = \{\lambda : \lambda \in \mathbb{C}, |\lambda - z_0| < \delta\}$ disjoint with $\sigma(A)$, the operator K defined by

$$K = F^{-1}(A^* - \bar{z}_0 I)^{-1} C^*,$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n} (A^* - \bar{z}_0 I)^{-n-1} C^* C (A - z_0 I)^{-n-1},$$

has the property that $\sigma(A - KC) \subset \Delta = \{\lambda : \lambda \in \mathbb{C}, |\lambda - z_0| < \delta\}$

LEMMA 2.4 (Theorem A.1 of [1]). *Let $A \in L(X)$, $C \in L(X, Y)$ be bounded linear operators acting between complex Banach spaces. Then the operator*

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \in L(X, Y^m)$$

is left invertible for some integer $m > 0$ if and only if the operator

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \in L(X, X + Y)$$

is left invertible for each $\lambda \in \mathbb{C}$.

THEOREM 2.5. *Assume $A \in M_n(W)$, $C \in M_{r \times n}(W)$. Then the observed system (C, A) is exactly observable if and only if there exists an operator $K \in M_{n \times r}(W)$ such that $\sigma(A) \cap \sigma(A - KC) = \varnothing$.*

Proof. Necessity: If $\sigma(A) \cap \sigma(A - KC) = \varnothing$, then from Lemma 1.5, for every $\varphi \in X$,

$$\sigma\big(\hat{A}(\varphi)\big) \cap \sigma\big(\hat{A}(\varphi) - \hat{K}(\varphi)\hat{C}(\varphi)\big) = \varnothing.$$

Using Theorem 2.3 and Lemma 2.4, we have that $(\hat{C}(\varphi), \hat{A}(\varphi))$ is observable for every $\varphi \in X$. By Lemma 2.1, (C, A) is exactly observable.

Sufficiency: If (C, A) is exactly observable, take same open set Ω in $\mathbb{C}$ such that $\sigma(A) \subset \Omega$. Choose some disc $\Delta = \{\lambda : |\lambda - z_0| < \delta,\ \lambda \in \mathbb{C}\}$ such that $\Omega \cap \Delta = \varnothing$. So the operator $A - z_0 I$ is invertible, and the spectral radius of $(A - z_0 I)^{-1}$ is less than δ^{-1}. It follows that the serices

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A^* - \bar{z}_0 I)^{-n-1} C^* C (A - z_0 I)^{-n-1}$$

converges in operator norm; hence F is well defined.

For every $\varphi \in X$

$$\hat{F}(\varphi) = \sum_{n=0}^{\infty} \delta^{2n}\big[\hat{A}^*(\varphi) - \bar{z}_0 I\big]^{-n-1} \hat{C}^*(\varphi)\hat{C}(\varphi)\big[\hat{A}(\varphi) - z_0 I\big]^{-n-1}.$$

Because (C, A) is exactly observable, we have that $(\hat{C}(\varphi), \hat{A}(\varphi))$ is observable for every $\varphi \in X$, and $\hat{F}(\varphi) > 0$ (strictly) (this can be seen in the process of the proof of Theorem 2.1 in [1]), so $\hat{F}(\varphi)$ is invertible. But

$$\sigma(F) = \cup \sigma\big(\hat{F}(\varphi)\big);$$

thus F is invertible.

It is easy to see that

$$\delta^2 F - (A^* - \bar{z}_0 I) F (A - z_0 I) = -C^* C.$$

Let

$$K = F^{-1}(A^* - \bar{z}_0 I)^{-1} C^*,$$

$$A - KC = z_0 I + \delta^2 F^{-1}(A^* - \bar{z}_0 I) F,$$

$$\lambda I - (A - KC) = F^{-1}\big[(\lambda - z_0) I - \delta^2(A^* - \bar{z}_0 I)\big]^{-1} F.$$

Since the spectral radius of $(A^* - \bar{z}_0 I)^{-1}$ is less than δ^{-1}, the operator

$$\left[(\lambda - z_0)I - \delta^2(A^* - \bar{z}_0 I)\right]^{-1}$$

is invertible whenever $|\lambda - z_0| \geqslant \delta$. It follows that

$$\sigma(A - KC) \subset \{\lambda : \lambda \in \mathbb{C}, |\lambda - z_0| < \delta\},$$

so

$$\sigma(A) \cap \sigma(A - KC) = \varnothing. \qquad \blacksquare$$

COROLLARY 2.6. *If (C, A) is exactly observable, and $\Delta \neq \varnothing$ is any open set in $\mathbb{C}$, then there exists an operator $K \in M_{n \times r}(W)$ such that $\sigma(A - KC) \subset \Delta$.*

Proof. Without loss of generality we may assume that $\mathbb{C} \setminus [\Delta \cup \sigma(A)]$ has a nonempty interior. Choose an open disc $|\lambda - z_0| < \delta$ which is disjoint with Δ and $\sigma(A)$. Since (C, A) is exactly observable, i.e., the operator $K_m(C, A)$ is left invertible for some positive integer m, we can apply the second part of the proof of Theorem 2.5 to show that there exists $K_1 \in M_{n \times r}(W)$ such that

$$\sigma(A - K_1 C) \subset \{\lambda : |\lambda - z_0| < \delta\}.$$

Now observe that $\sigma(A - K_1 C)$ is disjoint with Δ. Further, from our hypothesis it follows that the operator $K_m(C, A - K_1 C)$ is also left invertible for some m. Apply the second part of Theorem 2.5 to the pair $(C, A - K_1 C)$, so there is an operator $K_2 \in L(Y, X)$ such that $\sigma(A - K_1 C - K_2 C) \subset \Delta$. Hence, the operator $K = K_1 + K_2$ has the desired properties. $\blacksquare$

REMARK 1. Obviously, Theorem 4.1 of [1] and Theorem 2.5 have the same form, but they are different results and cannot be contained in each other, because the acting spaces are different and cannot be contained in each other.

REMARK 2. In the finite dimensional and Hilbert cases, it has been proved that exact observability or controllability and spectrum assignment are equivalent, i.e., (C, A) $[(A, B)]$ is exactly observable [controllable] if and only if for every compact set Λ there exists an operator K $[F]$ such that

$\sigma(A + KC)$ $[\sigma(A + BF)] = \Lambda$; see [2, 5]. We guess that this is right in the case of a commutative Hermitian Banach * algebra as well. If so, Corollary 2.6 is a natural corollary of the result.

LEMMA 2.7 (Theorem A.2 of [1]). *Let $A \in L(X)$, $B \in L(Y, X)$ be bounded operators acting between complex Banach spaces. Then the operator*

$$\begin{bmatrix} B & AB & \cdots & A^{m-1}B \end{bmatrix} \in L(U^m, X)$$

is right invertible for some $m > 0$ if and only if the operator

$$[\lambda I - A \quad B] \in L(X + U, X)$$

is right invertible for every λ in $\mathbb{C}$.

Dually, the following statements hold.

THEOREM 2.8. *Assume $A \in M_{n\times n}(W)$, $B \in M_{n\times m}(W)$. The control system (A, B) is exactly controllable if and only if there exists an operator $F \in M_{m\times n}(W)$ such that $\sigma(A) \cap \sigma(A - BF) = \varnothing$.*

COROLLARY 2.9. *If (A, B) is exactly controllable, and $\Delta \neq \varnothing$ is any open set in $\mathbb{C}$, then there exists an operator $F \in M_{m\times n}(W)$ such that $\sigma(A - BF) \subset \Delta$.*

III. STABILITY

We quote Lyapunov's theorem from [2, 3], and proceed to apply it in our context, namely where $M_n(W)$ replaces $M_n(\mathbb{C})$.

THEOREM 3.1. *Let $A \in M_n(\mathbb{C})$. If there are positive definite operators P and Q in $M_n(\mathbb{C})$, with Q positive definite, such that*

$$A^*P + PA = -Q, \tag{3.1}$$

then $\sigma(A)$ is contained in the open left half plane; conversely, if $\sigma(A)$ is contained in the left half plane, then for every positive definite $Q \in M_n(\mathbb{C})$, there is a unique positive definite P satisfying the equation (3.1).

THEOREM 3.2. *Let $A \in M_n(W)$. If there are positive definite operators P and Q in $M_n(W)$ such that*

$$A^*P + PA = -Q, \tag{3.2}$$

then $\sigma(A)$ is contained in $\mathbb{C}_-$.

Conversely, if $\sigma(A)$ is contained in the open left half plane, then for every positive definite $Q \in M_n(W)$, there is a positive definite P satisfying the equation (3.2).

Proof. Assume $A^*P + PA = -Q$, where Q and P are positive definite operators in $M_n(W)$. Applying the Gelfand transformation to the above equality, then

$$\hat{A}(\varphi)^* \hat{P}(\varphi) + \hat{P}(\varphi)\hat{A}(\varphi) = -\hat{Q}(\varphi)$$

for every $\varphi \in X$. By Lemma 1.6, since $Q > 0$, $P > 0$, we have

$$\hat{Q}(\varphi) > 0, \hat{P}(\varphi) > 0 \qquad \text{for every} \quad \varphi \in X;$$

by Theorem 3.1,

$$\sigma\big(\hat{A}(\varphi)\big) \subset \mathbb{C}_-,$$

and

$$\sigma(A) = \bigcup_{\varphi \in X} \sigma\big(\hat{A}(\varphi)\big) \qquad \text{(Lemma 1.5)},$$

so

$$\sigma(A) \subset \mathbb{C}_-.$$

On the other hand, if $\sigma(A) \subset \mathbb{C}_-$, given $Q > 0$, define

$$P = \int_0^\infty e^{A^*t} Q e^{At}\, dt.$$

Then it can be shown that the integral above is norm convergent and $P > 0$, and P satisfies the equation (3.1).

From $\sigma(F) \subset \mathbb{C}_-$, there exists a constant M, $M \geqslant 1$, and $\delta > 0$ such that

$$\|e^{Ft}\| \leqslant Me^{-\delta t}, \qquad t > 0.$$

Let

$$P_n = \int_0^n e^{A^*t} Q e^{At}\, dt.$$

Then, if $n < m$,

$$\|P_m - P_n\| = \left\| \int_n^m e^{A^*t} Q e^{At}\, dt \right\|$$

$$\leqslant \int_n^m \| e^{A^*t} Q e^{At} \|\, dt$$

$$\leqslant \|Q\| MH \int_n^m c^{-\delta t}\, dt,$$

where H is a constant. It follows that P_n is a norm Cauchy sequence, so the improper integral defining P is norm convergent, and

$$A^*P + PA = \int_0^\infty e^{A^*t}(A^*Q + QA) e^{At}\, dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^*t} Q e^{At}\right) dt$$

$$= -Q$$

For every $\varphi \in X$

$$\hat{P}(\varphi) = \int_0^\infty e^{\hat{A}^*(\varphi)t} \hat{Q}(\varphi) e^{\hat{A}(\varphi)t}\, dt.$$

Because $Q > 0$, then

$$\hat{Q}(\varphi) > 0, \qquad \hat{P}(\varphi) > 0;$$

hence, $P > 0$. This completes the proof of Theorem 3.2. ■

COROLLARY 3.3. *Suppose W is semisimple. If $\dot{x}(t) = Ax(t)$ is stable, then the operator equation* (3.2) *has a unique solution.*

IV. STABILIZABILITY AND DETECTABILITY

THEOREM 4.1. *Let $A \in M_n(W)$, $B \in M_{n\times m}$. If for some positive definite operator Q the algebraic Riccati equation*

$$A^*P + PA - PBB^*P = -Q$$

has a positive definite solution P, the $\sigma(A - BB^*P)$ is contained in the open left half plane, i.e., (A, B) is stabilizable

Proof. From the assumpotion, we can get

$$\begin{aligned}(A - BB^*P)^*P + P(A - BB^*P) &= A^*P + PA - 2PBB^*P \\ &= -(Q + PBB^*P).\end{aligned}$$

Because $Q > 0$, then for every $\varphi \in X$

$$\begin{aligned}\hat{Q}(\varphi) + \hat{P}(\varphi)\hat{B}(\varphi)\hat{B}(\varphi)^*P(\varphi) &> 0, \\ Q + PBB^*P &> 0.\end{aligned}$$

By Theorem 3.2,

$$\sigma(A - BB^*P) \subset \mathbb{C}_-. \qquad \blacksquare$$

On the other hand, if W is a commutative C^* algebra, the converse of Theorem 4.1 is correct.

THEOREM 4.2. *Let W be a commutative C^* algebra, $A \in M_n(W)$, $B \in M_{n\times m}(W)$. If there exists an $F \in M_{m\times n}(W)$ such that $\sigma(A - BF) \subset \mathbb{C}_-$, i.e., (A, B) is stabilizable, then for every positive definite $Q \in M_n(W)$, the algebraic Riccati equation has a positive definite solution P in $M_n(W)$.*

Proof. Because (A, B) is stabilizable, then from Theorem 3.6 of [18], the algebraic Riccati equation

$$\hat{A}^*(\varphi)P(\varphi) + P(\varphi)\hat{A}(\varphi) - P(\varphi)\hat{B}(\varphi)\hat{B}^*(\varphi)P(\varphi) + 1 = 0$$
$$(\varphi \in X)$$

has a continuous positive definite solution $P(\varphi) \in M_n(\mathbb{C})$; thus the algebraic Riccati equation

$$\hat{A}^*(\varphi)P(\varphi) + P(\varphi)\hat{A}(\varphi) - P(\varphi)\hat{B}(\varphi)\hat{B}^*(\varphi)P(\varphi) = -\hat{Q}(\varphi)$$
$$(\varphi \in X)$$

has a continuous positive definite solution $P(\varphi) \in M_n(\mathbb{C})$. But W is a C^* algebra, so then there exists an element $S \in W$ [17] such that

$$\hat{S}(\varphi) = P(\varphi),$$

i.e., S is positive definite. Thus the algebraic Riccati equation

$$A^*P + PA - PBB^*P = -Q$$

has a positive definite solution S. ■

LEMMA 4.3. *Given two natural numbers p, q with $p \geqslant q$, let $Q \in M_{q \times p}(W)$. Then the following conditions are equivalent:*

(1) *There exists $H \in M_{p \times q}(W)$ such that $QH = I_q$.*

(2) *For each $\varphi \in X$, $\hat{Q}(\varphi)$ considered as a linear transformation from $\mathbb{C}^q$ to $\mathbb{C}^q$ is onto.*

(3) *For each $\varphi \in X$, the rank of the scalar matrix $\hat{Q}(\varphi)$ is q.*

THEOREM 4.4 (Theorem 3.1 of [6]). *Given $A \in M_n(W)$, $B \in M_{n \times m}(W)$, the control system (A, B) is stabilizable if and only if for each $\varphi \in X$, the finite control system $(\hat{A}(\varphi), \hat{B}(\varphi))$ is stabilizable.*

THEOREM 4.5. *Given $A \in M_n(W)$, $B \in M_{n \times m}(W)$, the control system (A, B) is stabilizable if and only if $[\lambda I - A \;\; B]$ is right invertible for every $\lambda \in \mathbb{C}$ with* $\operatorname{Re} \lambda \geqslant 0$.

Proof. Necessity: If (A, B) is stabilizable, then from Theorem 4.4, for every $\varphi \in X$, $(\hat{A}(\varphi), \hat{B}(\varphi)$ is stabilizable, i.e., $[\lambda I - \hat{A}(\varphi) \;\; \hat{B}(\varphi)]$ is right invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$. So $[\lambda I - A \;\; B]$ is right invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$.

Sufficiency: If $[\lambda I - A \;\; B]$ is right invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$, then for every $\varphi \in X$, $[\lambda I - \hat{A}(\varphi) \;\; \hat{B}(\varphi)]$ is invertible for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geqslant 0$. Thus, $[\hat{A}(\varphi) \;\; \hat{B}(\varphi)]$ is stabilizable. Using Theorem 4.4, we have that (A, B) is stabilizable. ■

THEOREM 4.6. *Assume $A \in M_n(W)$, $C \in M_{r \times n}(W)$. Then the observed system (C, A) is detectable if and only if (A^*, C^*) is stabilizable.*

Proof. Assume (C, A) is detectable. Then there exists a $K \in M_{n \times r}(W)$ such that

$$\sigma(A + KC) \subset \mathbb{C}_-.$$

But

$$\sigma(A^* + C^*K^*) = \sigma((A + KC)^*) = \{\bar{\lambda} : \lambda \in \sigma(A + KC)\},$$

so

$$\sigma(A^* + C^*K^*) \subset \mathbb{C}_-,$$

i.e., (A^*, C^*) *is stabilizable.* ■

THEOREM 4.7. (C, A) *is detectable if and only if*

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

is left invertible for every $\lambda \in \mathbb{C}$ *with* Re $\lambda \geqslant 0$.

Proof. It is easy to see that

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$$

is left invertible for every $\lambda \in \mathbb{C}$ with $\lambda \geqslant 0$ if and only if

$$[\lambda I - A^* \quad C^*]$$

is right invertible for every $\lambda \in \mathbb{C}$ with Re $\lambda \geqslant 0$. Then from Theorem 4.6 the result can easily be obtained. ■

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REFERENCES

1 M. K. Kaashoek, C. V. M. van der Mee, and L. Rodman, Analytic operator functions with compact spectrum. III. Hilbert space case: Inverse problem and applications, *J. Oper. Theory* 10:219–250 (1983).

2 W. M. Wonham, *Linear Multivariable Control, a Geometric Approach*, Springer-Verlag, New York, 1979.

3 W. A. Wolovich, *Linear Multivariable Systems*, Appl. Math Sci. 11, Springer-Verlag, 1974.

4 D. J. Clements and K. Glover, Spectral factorization via Hermitian pencills, *Linear Algebra Appl.* 122/123/124:797–846 (1989).
5 K. Takahashi, Exact controllability and spectrum assignment, *J. Math. Anal. Appl.* 104:537–547 (1984).
6 Chen Wanyi, Stabilizability, controllability and observability of linear continuous-time systems defined over a commutative Banach algebra, *Linear Algebra Appl.* 144:1–10 (1991).
7 E. W. Kamen and W. L. Green, Asymptotic stability of linear difference equations defined over a commutative Banach algebra, *J. Math. Anal. Appl.* 75(2):584–601 (1980).
8 ____, Stabilizability of linear systems over a commutative normed algebra with application to spatially-distributed and parameter-independent systems, *SIAM J. Control* 23(1):1–18 (1985).
9 ____, Stabilizability of linear discrete-time systems defined over a commutative normed algebra, in *Proceedings of the 19th IEEE Conference on Decision and Control*, Albuquerque, N.M., 1980, pp. 264–268.
10 ____, Addendum to "Asymptotic stability of linear difference equation defined over a commutative Banach algebra," *J. Math. Anal. Appl.* 84:25–33 (1981).
11 E. W. Kamen, Linear discrete-time systems over a commutative Banach algebra with applications to two-dimensional systems, in *Algebraic and Geometric Methods in Linear Systems Theory*, Lectures in Appl. Math. 18 (C. I. Byrnes and C. Martin, Eds.), Amer. Math. Soc., Providence, 1980, pp. 225–237.
12 C. I. Byrnes, Realization theory and quadratic optimal controllers for systems defined over Banach and Fréchet algebras, in *Proceedings of the 19th IEEE Conference on Decision and Control*, Albuquerque, N.M., 1980, pp. 247–251.
13 R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear System Theory*, Springer-Verlag, New York, 1978.
14 H. J. Zwart, *Geometric Theory for Infinite Dimensional Systems*, Lecture Notes in Control and Inform. Sci. 115, Springer-Verlag, Berlin, 1989.
15 P. Fuhrmann, *Linear Systems and Operators in Hilbert Space*, McGraw-Hill, New York, 1981.
16 ____, Exact controllability and observability of realization theory in Hilbert space, *J. Math. Anal. Appl.*, 53:377–392 (1976).
17 J. Dixmier, *C^*-Algebra*, North-Holland, Amsterdam, 1977.
18 Hwang Ling and Zhu Wei Ling, The small disturbance problem associated with the algebraic Riccati equation of continuous linear time-invariant systems (in Chinese; English abstract), *Appl. Math. Mech.* 3(5):653–660 (1982).
19 T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.

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